5-2 Verifying Trigonometric Identities

Verify each identity.

2. $\sec^2 \theta (1 - \cos^2 \theta) = \tan^2 \theta$

ANSWER:

$$\begin{aligned}\sec^2\theta(1-\cos^2\theta) &= \sec^2\theta - \sec^2\theta\cos^2\theta \\ &= \sec^2\theta - 1 \\ &= \tan^2\theta\end{aligned}$$

4. $\csc \theta - \cos \theta \cot \theta = \sin \theta$

ANSWER:

$$\begin{aligned}\csc\theta - \cos\theta\cot\theta &= \frac{1}{\sin\theta} - \cos\theta\left(\frac{\cos\theta}{\sin\theta}\right) \\ &= \frac{1-\cos^2\theta}{\sin\theta} \\ &= \frac{\sin^2\theta}{\sin\theta} \\ &= \sin\theta\end{aligned}$$

6. $\tan \theta \csc^2 \theta - \tan \theta = \cot \theta$

ANSWER:

$$\begin{aligned}\tan\theta\csc^2\theta - \tan\theta &= \tan\theta(\csc^2\theta - 1) \\ &= \tan\theta\cot^2\theta \\ &= \frac{\sin\theta}{\cos\theta}\cdot\frac{\cos^2\theta}{\sin^2\theta} \\ &= \frac{\cos\theta}{\sin\theta} \\ &= \cot\theta\end{aligned}$$

8. $\frac{\sin\theta}{1-\cos\theta}+\frac{1-\cos\theta}{\sin\theta}=2\csc\theta$

ANSWER:

$$\frac{\sin\theta}{1-\cos\theta}+\frac{1-\cos\theta}{\sin\theta}$$
$$=\frac{\sin\theta}{\sin\theta}\cdot\frac{\sin\theta}{1-\cos\theta}+\frac{1-\cos\theta}{1-\cos\theta}\cdot\frac{1-\cos\theta}{\sin\theta}$$
$$=\frac{\sin^2\theta}{\sin\theta(1-\cos\theta)}+\frac{1-2\cos\theta+\cos^2\theta}{\sin\theta(1-\cos\theta)}$$
$$=\frac{\sin^2\theta+\cos^2\theta+1-2\cos\theta}{\sin\theta(1-\cos\theta)}$$
$$=\frac{2-2\cos\theta}{\sin\theta(1-\cos\theta)}$$
$$=\frac{2(1-\cos\theta)}{\sin\theta(1-\cos\theta)}$$
$$=\frac{2}{\sin\theta}$$
$$=2\csc\theta$$

10. $\frac{\sin\theta}{1-\cot\theta}+\frac{\cos\theta}{1-\tan\theta}=\sin\theta+\cos\theta$

ANSWER:

$$\frac{\sin\theta}{1-\cot\theta}+\frac{\cos\theta}{1-\tan\theta}$$
$$=\frac{\sin\theta}{1-\frac{\cos\theta}{\sin\theta}}+\frac{\cos\theta}{1-\frac{\sin\theta}{\cos\theta}}$$
$$=\frac{\sin\theta}{\frac{\sin\theta}{\sin\theta}-\frac{\cos\theta}{\sin\theta}}+\frac{\cos\theta}{\frac{\cos\theta}{\cos\theta}-\frac{\sin\theta}{\cos\theta}}$$
$$=\frac{\sin^2\theta}{\sin\theta-\cos\theta}+\frac{\cos^2\theta}{\cos\theta-\sin\theta}$$
$$=\frac{\sin^2\theta}{\sin\theta-\cos\theta}-\frac{\cos^2\theta}{\sin\theta-\cos\theta}$$
$$=\frac{\sin^2\theta-\cos^2\theta}{\sin\theta-\cos\theta}$$
$$=\frac{(\sin\theta+\cos\theta)(\sin\theta-\cos\theta)}{\sin\theta-\cos\theta}$$
$$=\sin\theta+\cos\theta$$

12. $\frac{1}{\csc\theta+1}+\frac{1}{\csc\theta-1}=2\sec^2\theta\sin\theta$

ANSWER:

$$\frac{1}{\csc\theta+1}+\frac{1}{\csc\theta-1}$$
$$=\frac{\csc\theta-1}{\csc\theta-1}\cdot\frac{1}{\csc\theta+1}+\frac{\csc\theta+1}{\csc\theta+1}\cdot\frac{1}{\csc\theta-1}$$
$$=\frac{\csc\theta-1}{\csc^2\theta-1}+\frac{\csc\theta+1}{\csc^2\theta-1}$$
$$=\frac{2\csc\theta}{\csc^2\theta-1}$$
$$=\frac{2\csc\theta}{\cot^2\theta}$$
$$=\frac{2\left(\frac{1}{\sin\theta}\right)}{\frac{\cos^2\theta}{\sin^2\theta}}$$
$$=\frac{2}{\sin\theta}\cdot\frac{\sin^2\theta}{\cos^2\theta}$$
$$=\frac{2\sin\theta}{\cos^2\theta}$$
$$=\left(\frac{2}{\cos^2\theta}\right)\sin\theta$$
$$=2\sec^2\theta\sin\theta$$

14. $\cos^4\theta-\sin^4\theta=\cos^2\theta-\sin^2\theta$

ANSWER:

$$\cos^4\theta-\sin^4\theta=(\cos^2\theta+\sin^2\theta)(\cos^2\theta-\sin^2\theta)$$
$$=\cos^2\theta-\sin^2\theta$$

16. $\frac{\cos\theta}{1+\sin\theta}+\frac{\cos\theta}{1-\sin\theta}=2\sec\theta$

ANSWER:

$$\frac{\cos\theta}{1+\sin\theta}+\frac{\cos\theta}{1-\sin\theta}$$
$$=\frac{\cos\theta}{1+\sin\theta}\cdot\frac{1-\sin\theta}{1-\sin\theta}+\frac{\cos\theta}{1-\sin\theta}\cdot\frac{1+\sin\theta}{1+\sin\theta}$$
$$=\frac{\cos\theta(1-\sin\theta)+\cos\theta(1+\sin\theta)}{(1+\sin\theta)(1-\sin\theta)}$$
$$=\frac{\cos\theta-\sin\theta\cos\theta+\cos\theta+\sin\theta\cos\theta}{1-\sin^2\theta}$$
$$=\frac{2\cos\theta}{\cos^2\theta}$$
$$=\frac{2}{\cos\theta}$$
$$=2\sec\theta$$

18. $\frac{\csc^2\theta+2\csc\theta-3}{\csc^2\theta-1}=\frac{\csc\theta+3}{\csc\theta+1}$

ANSWER:

$$\frac{\csc^2\theta+2\csc\theta-3}{\csc^2\theta-1}=\frac{(\csc\theta+3)(\csc\theta-1)}{(\csc\theta+1)(\csc\theta-1)}$$
$$=\frac{\csc\theta+3}{\csc\theta+1}$$

Verify each identity.

20. $(\csc\theta+\cot\theta)(1-\cos\theta)=\sin\theta$

ANSWER:

$$(\csc\theta+\cot\theta)(1-\cos\theta)$$
$$=\csc\theta-\csc\theta\cos\theta+\cot\theta-\cot\theta\cos\theta$$
$$=\frac{1}{\sin\theta}-\left(\frac{1}{\sin\theta}\right)\cos\theta+\left(\frac{\cos\theta}{\sin\theta}\right)-\left(\frac{\cos\theta}{\sin\theta}\right)\cos\theta$$
$$=\frac{1}{\sin\theta}-\frac{\cos\theta}{\sin\theta}+\frac{\cos\theta}{\sin\theta}-\frac{\cos^2\theta}{\sin\theta}$$
$$=\frac{1-\cos^2\theta}{\sin\theta}$$
$$=\frac{\sin^2\theta}{\sin\theta}$$
$$=\sin\theta$$

5-2 Verifying Trigonometric Identities

22. $\dfrac{1-\tan^2\theta}{1-\cot^2\theta} = \dfrac{\cos^2\theta - 1}{\cos^2\theta}$

ANSWER:

$$\frac{1-\tan^2\theta}{1-\cot^2\theta}$$

$$= \frac{1-\frac{\sin^2\theta}{\cos^2\theta}}{1-\frac{\cos^2\theta}{\sin^2\theta}}$$

$$= \frac{\frac{\cos^2\theta}{\cos^2\theta} - \frac{\sin^2\theta}{\cos^2\theta}}{\frac{\sin^2\theta}{\sin^2\theta} - \frac{\cos^2\theta}{\sin^2\theta}}$$

$$= \frac{\frac{\cos^2\theta-\sin^2\theta}{\cos^2\theta}}{\frac{\sin^2\theta-\cos^2\theta}{\sin^2\theta}}$$

$$= \frac{\cos^2\theta-\sin^2\theta}{\cos^2\theta} \cdot \frac{\sin^2\theta}{\sin^2\theta-\cos^2\theta}$$

$$= \frac{\cos^2\theta-(1-\cos^2\theta)}{\cos^2\theta} \cdot \frac{1-\cos^2\theta}{(1-\cos^2\theta)-\cos^2\theta}$$

$$= \frac{-1+2\cos^2\theta}{\cos^2\theta} \cdot \frac{1-\cos^2\theta}{1-2\cos^2\theta}$$

$$= \frac{-(1-2\cos^2\theta)\left(1-\cos^2\theta\right)}{\cos^2\theta\left(1-2\cos^2\theta\right)}$$

$$= \frac{-(1-\cos^2\theta)}{\cos^2\theta}$$

$$= \frac{\cos^2\theta-1}{\cos^2\theta}$$

24. $(\csc\ \theta\ -\ \cot\ \theta)^2 = \dfrac{1-\cos\theta}{1+\cos\theta}$

ANSWER:

$$\begin{aligned}(\csc\theta-\cot\theta)^2 &= \csc^2\theta-2\csc\theta\cot\theta+\cot^2\theta\\ &= \frac{1}{\sin^2\theta}-\frac{2}{\sin\theta}\cdot\frac{\cos\theta}{\sin\theta}+\frac{\cos^2\theta}{\sin^2\theta}\\ &= \frac{1-2\cos\theta+\cos^2\theta}{\sin^2\theta}\\ &= \frac{1-2\cos\theta+\cos^2\theta}{1-\cos^2\theta}\\ &= \frac{(1-\cos\theta)^2}{(1+\cos\theta)(1-\cos\theta)}\\ &= \frac{1-\cos\theta}{1+\cos\theta}\end{aligned}$$

26. $\tan^2\ \theta\ \cos^2\ \theta\ = 1\ -\ \cos^2\ \theta$

ANSWER:

$$\begin{aligned}\tan^2\theta\cos^2\theta &= \frac{\sin^2\theta}{\cos^2\theta}\cdot\cos^2\theta\\ &= \sin^2\theta\\ &= 1-\cos^2\theta\end{aligned}$$

28. $1\ -\ \tan^4\ \theta = 2\ \sec^2\ \theta\ -\ \sec^4\ \theta$

ANSWER:

$$\begin{aligned}1-\tan^4\theta &= (1-\tan^2\theta)(1+\tan^2\theta)\\ &= [1-(\sec^2\theta-1)](\sec^2\theta)\\ &= (2-\sec^2\theta)(\sec^2\theta)\\ &= 2\sec^2\theta-\sec^4\theta\end{aligned}$$

30. $\frac{1+\tan\theta}{\sin\theta+\cos\theta} = \sec\theta$

ANSWER:

$$\begin{aligned}\frac{1+\tan\theta}{\sin\theta+\cos\theta} &= \frac{1+\frac{\sin\theta}{\cos\theta}}{\sin\theta+\cos\theta}\\ &= \frac{\frac{\cos\theta}{\cos\theta}+\frac{\sin\theta}{\cos\theta}}{\sin\theta+\cos\theta}\\ &= \frac{\frac{\cos\theta+\sin\theta}{\cos\theta}}{\sin\theta+\cos\theta}\\ &= \frac{\cos\theta+\sin\theta}{\cos\theta}\cdot\frac{1}{\sin\theta+\cos\theta}\\ &= \frac{1}{\cos\theta}\\ &= \sec\theta\end{aligned}$$